Name: ....

Form:........................

ASCHAM SCHOOL
MATHEMATICS EXAMINATION
FORM 6 - 3 UNIT
1999

July 1999 Time allowed: 2 hours

* All questions should be attempted
* All necessary working must be shown
* All questions are of equal value
* Marks may not be awarded for careless or badly arranged work.
* Write your name on each booklet clearly marked:
  Question 1, Question 2, ... etc.
* Begin each question in a new booklet.
* Approved calculators may be used.
* Copies of diagrams for all questions are provided on pages 11-14 in order to save time. You may use them but **you must staple them into your booklets**.

Question 1 Marks:

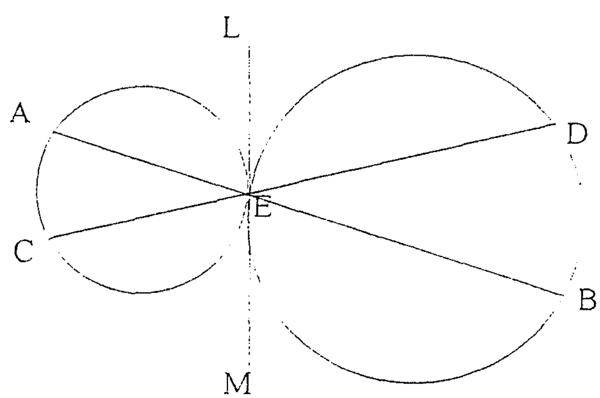
(a) Find the acute angle, to the nearest degree, between the lines $y = 3x + 1$ and $y = -x + 6$ **3**

(b) Solve the inequality $\frac{1}{x+1} < 3, \quad x \neq -1$ **3**

(c) Find the coordinates of the point P which divides the interval AB with end points A(-1, 2) and B(3, -5) internally in the ratio 2:3. **2**

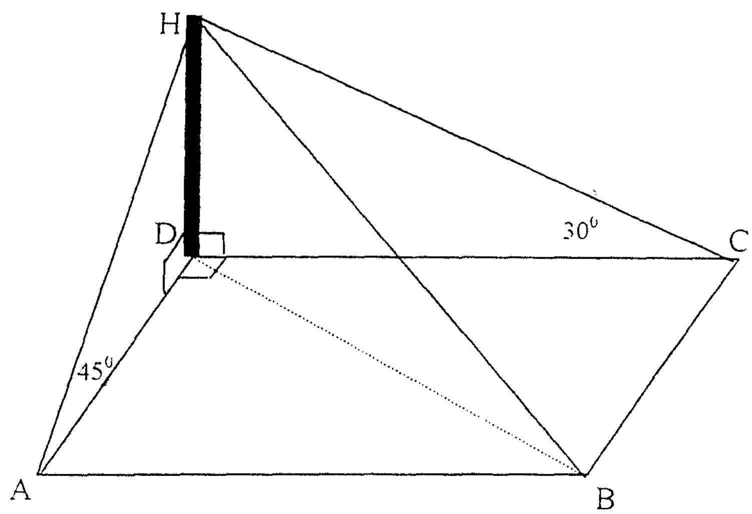
(d) Use the substitution $u = t + 1$ to evaluate $\int_0^1 \frac{t}{\sqrt{t+1}}\, dt$ **3**

(e) Two circles touch externally at E. **3**



(A copy of the diagram above is on page 10.)
AB and CD intersect at E. LM is a common tangent at E
Prove that AC is parallel to DB.

Question 2 **Marks:**



(a) A post HD stands vertically at one corner of a rectangular field ABCD. The angles of elevation of the top H of the post from the nearest corners A and C respectively are 30^0 and 45^0.
(A copy of the diagram above is on page 13.)

(i) If $AD = a$ units, find the length of BD in terms of a. 2

(ii) Hence find the angle of elevation of H from the corner B to the nearest minute. 1

(b) Taking $x = -\frac{\pi}{6}$ as a first approximation to the root of the equation $2x + \cos x = 0$, use Newton's method once to show that a better approximation to the root of the equation is $\frac{-\pi - 6\sqrt{3}}{30}$ 4

(c) (i) Find the domain and range of $f^{-1}(x) = \sin^{-1}(3x - 1)$ 2

(ii) Sketch the graph of $y = f^{-1}(x)$. 2

(iii) Find the equation representing the inverse function $f(x)$ and state the domain and range. 3

Question 3 **Marks:**

(a) (i) Express $3\sin x - \sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$, where $A > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. **3**

(ii) Determine the minimum value of $3\sin x - \sqrt{3}\cos x$. **1**

(iii) Solve $3\sin x - \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$. **3**

(b) Newton's Law of cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature. This rate can be expressed by the differential equation:

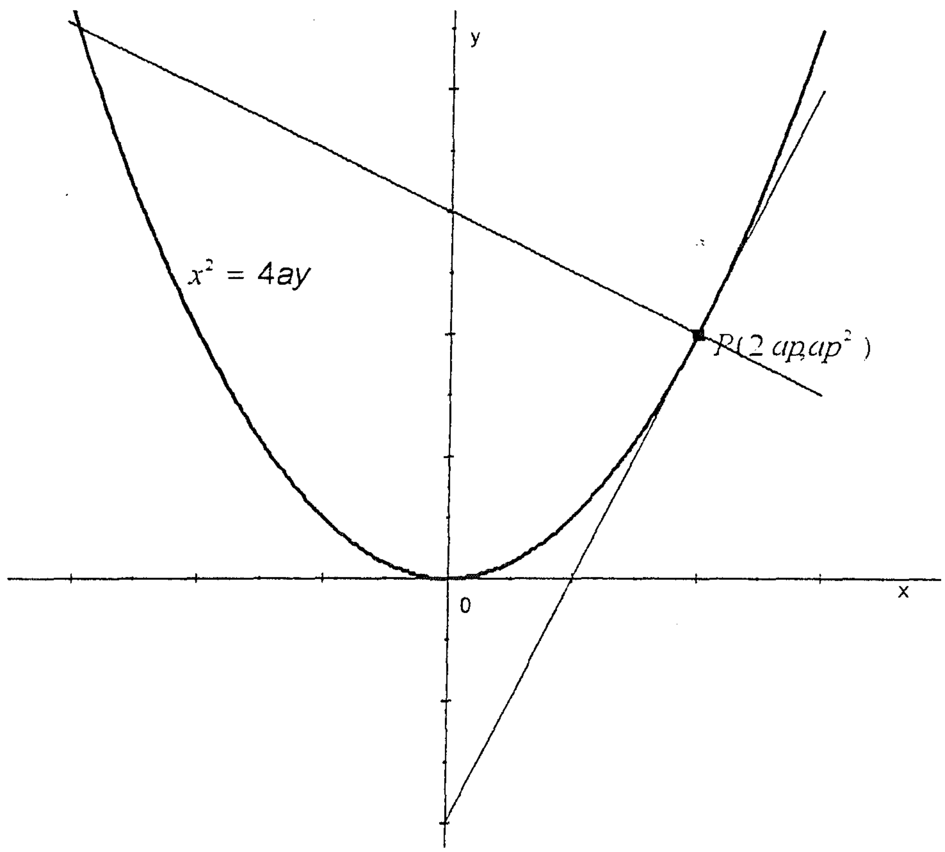
$$\frac{dT}{dt} = -k(T - T_0),$$

where T is the temperature of the body, T_0 is the temperature of the surroundings, t is the time in minutes and k is a constant.

(i) Show that $T = T_0 + Ae^{-kt}$, where A is a constant, is a solution of the differential equation $\frac{dT}{dt} = -k(T - T_0)$. **2**

(ii) A cup of tea cools from 85°C to 80°C in 1 minute at a room temperature of 25°C. Find the temperature of the cup of tea after a further 4 minutes have elapsed. Answer to the nearest degree. **5**

Question 4 **Marks:**



(a) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. **3**
Show the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$.

(b) Write down the equation of the normal to the parabola at Q. The normals intersect at N. Find the coordinates of N. **3**

(c) Show the equation of the chord PQ is $y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$ **3**

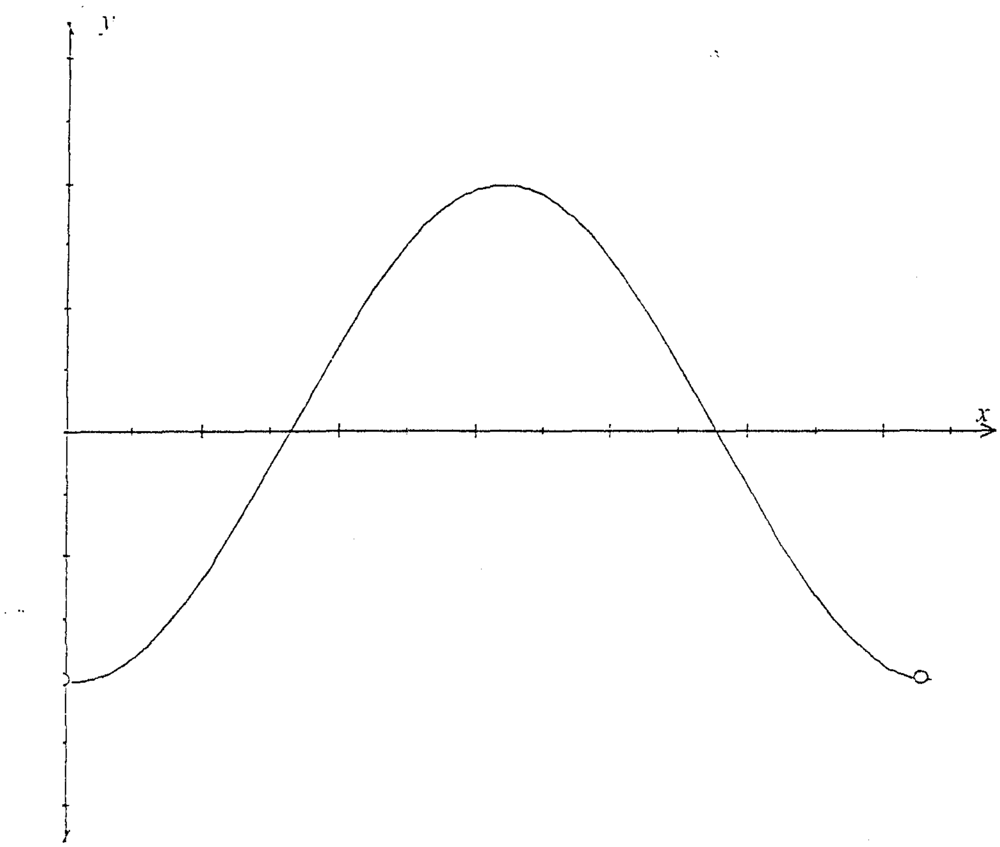
and determine the condition necessary for PQ to be a focal chord.

(d) If PQ is a focal chord and N is the intersection of the normals, find the equation of the locus of N. **4**

(e) (A copy of the diagram above is on page 11.) **1**
On the diagram above, the tangent and normal are drawn at P.
Mark clearly on your own diagram the points Q and N which correspond to P.

Question 5 **Marks:**

(a) The graph of $x = -a\cos nt$ for $0 \le t \le \frac{2\pi}{n}$ is drawn below. (A copy of the diagram above is on page 12.) Label axes and show intercepts accurately. **2**



(b) On a certain day the depth of water in a harbour at low tide at 4:30 am is 5 metres. At the following high tide at 10:45 am the depth is 15 metres. Assuming the rise and fall of the surface of the water to be simple harmonic, find between what times during the morning a ship may safely enter the harbour if the minimum depth of $12\frac{1}{2}$ metres of water is required. **6**

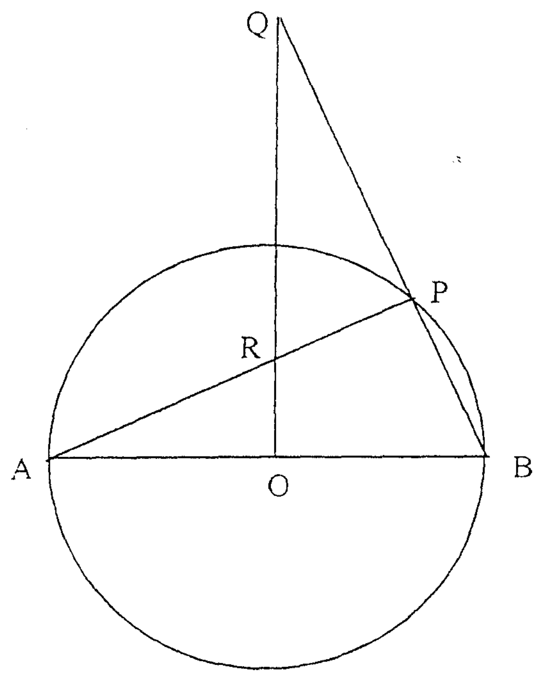
(c) Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(2-x)$ have values for $0 \le x \le \frac{\pi}{2}$

(i) show that $\sin(\sin^{-1} x - \cos^{-1} x) = 2x^2 - 1$ **3**

(ii) Hence, or otherwise, solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(2-x)$ **3**

Question 6 **Marks:**

(a) O is the centre of the circle. BPQ is a straight line ORQ is perpendicular to AOB as shown below.



(A copy of the diagram above is on page 14.)

Prove that:

(i) A, O, P, Q are concyclic, and **3**

(ii) $\angle OPA = \angle OQB$. **2**

(b) Prove by using mathematical induction that $5^n \geq 1 + 4n$, for $n > 1$, $n \in J^+$. **4**

(c) The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β, and γ. Evaluate

(i) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\alpha^2\beta^2\gamma + \beta^2\gamma^2\alpha + \alpha^2\gamma^2\beta$ **1**

(d) The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$. **2**

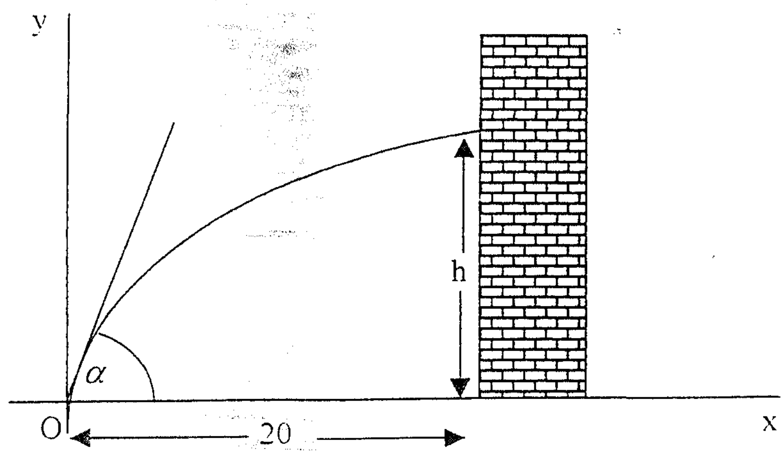
Using appropriate information from (c) above prove that

$$\sec a + \sec b + \sec c = 1$$

Question 7 **Marks:**

A softball player hits the ball from ground level with a speed of 20 ms^{-1} and an angle of elevation α. It flies toward a high wall 20 m away on level ground.

(a) Taking the origin at the point where the ball is hit. derive expressions for **3**



the horizontal and vertical components x and y of displacement at time t seconds. Take g = 10ms^{-2}.

(b) Hence find the equation of the path of the ball in flight in terms of x, y and. α. **1**

(c) Show that the height h at which the ball hits the wall is given by **2**

$$h = 20\tan\alpha - 5(1 + \tan^2\alpha).$$

(d) Using part (c) above, show that the maximum value of h occurs when $\tan\alpha = 2$. **2**

(e) Find **6**

(i) this maximum height h,

(ii) the speed and the angle at which the ball hits the wall in this case.

1999 3U TRIAL (ascham) (1)

$y = 3x + 2 \quad m_1 = 3 \qquad y = 1 - x \quad m_2 = -1$

$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ ✓

$= \left|\frac{3+1}{1+3(-1)}\right|$

$= 2$ ✓ $\qquad \therefore \alpha = 63^\circ 26'$

$= 63^\circ$ ✓ (nearest deg.)

$\frac{1}{x+1} < 3 \Rightarrow x+1 < 3(x+1)^2$ ✓

$= 3(x^2+2x+1)$

$3x^2 + 5x + 2 > 0$

$(3x+2)(x+1) > 0$ ✓

$x < -1$ ½ or $x > -\frac{2}{3}$ ½

2) $2:3 \quad 3, -5$

$P\left(\frac{2\times 3 - 3\times 1}{5}, \frac{2\times -5 + 3\times 2}{5}\right)$ ✓

$P\left(\frac{3}{5}, -\frac{4}{5}\right)$ ✓

$\int_0^1 \frac{t}{\sqrt{t+1}}\,dt \qquad u = t+1 \quad t = u-1 \quad du = dt \qquad$ if $t=0$, $u=1$; if $t=1$, $u=2$

$= \int_1^2 \frac{u-1}{\sqrt{u}}\,du$ ✓

$= \int_1^2 (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$

$= \left[\frac{2u^{3/2}}{3} - 2u^{1/2}\right]_1^2$ ✓

$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right)$

$= -\frac{2\sqrt{2}}{3} + \frac{4}{3}$ ✓

$= \frac{4 - 2\sqrt{2}}{3}$ (OR)

(2)

Q1. (e) $\angle AEL = \angle ACE$ (∠ in alt. segm.) ½

$\angle AEL = \angle MEB$ (vert. opp. ∠s) ½

$\angle BDE = \angle MEB$ (∠ in alt. segm) ½

$\therefore \angle ACE = \angle BDE$ ½

But $\angle ACE$ is alternate to $\angle BDE$ $\therefore AC \parallel DB$ ½

Q.2. (a)(i) In $\triangle ADH$ $\quad AD = DH = a$

In $\triangle HDC$ $\quad \tan 30^\circ = \frac{a}{DC}$

$\therefore DC = a\sqrt{3}$ ✓

In $\triangle BDC$ $\quad BD^2 = DC^2 + CB^2$

$= (a\sqrt{3})^2 + a^2$

$= 4a^2$

$\therefore BD = 2a$ ✓

(ii) In $\triangle HDB$ $\quad \frac{HD}{BD} = \tan H\hat{B}D$

$\frac{a}{2a} = \frac{1}{2} = \tan H\hat{B}D$

$\therefore \angle HBD = 26^\circ 34'$ (to nearest minute)

(b) $\frac{d}{dx}(2x + \cos x) = 2 - \sin x$ ✓

$f'(\frac{-\pi}{6}) = 2 + \sin\frac{\pi}{6}$

$= 2 + \frac{1}{2}$

$= \frac{5}{2}$ ½

$f(\frac{-\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$= \frac{3\sqrt{3} - 2\pi}{6}$ ½

$x_1 = -\frac{\pi}{6} - \frac{3\sqrt{3} - 2\pi}{6} \times \frac{2}{5}$ ✓

$= -\frac{\pi}{6} - \frac{3\sqrt{3}}{15} + \frac{2\pi}{15}$

$= \frac{-\pi - 6\sqrt{3}}{30}$ ✓

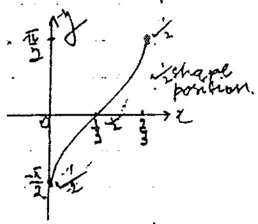
$y = \sin^{-1}(3x-1)$

Domain of $f^{-1}(x)$: $-1 \le 3x-1 \le 1$

$0 \le 3x \le 2$ ✓

$0 \le x \le \frac{2}{3}$ ✓

Range of $f^{-1}(x)$: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ ✓



$y = \sin^{-1}(3x-1)$ is the inverse function of $f(x)$

$\sin y = 3x - 1$ ✓ is the inverse of $f(x)$

$\sin x = 3y - 1$

$y = \frac{1}{3} + \frac{1}{3}\sin x$ is $f(x)$ ✓

so $f(x) = \frac{1}{3} + \frac{1}{3}\sin x$. Has domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and range $0 \le y \le \frac{2}{3}$ ✓

$3\sin x - \sqrt{3}\cos x = A\sin(x-\alpha)$
$= A\sin x\cos\alpha - A\cos x\sin\alpha$

$\frac{A\sin\alpha}{A\cos\alpha} = \frac{+\sqrt{3}}{3}$ $\therefore \tan\alpha = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$

$A^2\sin^2\alpha + A^2\cos^2\alpha = 9 + 3$ $\therefore A^2 = 12$, $A = 2\sqrt{3}$

$\therefore 3\sin x - \sqrt{3}\cos x = 2\sqrt{3}\sin(x - \frac{\pi}{6})$ ✓

Min. value $= -2\sqrt{3}$ ✓

$2\sqrt{3}\sin(x - \frac{\pi}{6}) = \sqrt{3}$ ✓

$\sin(x - \frac{\pi}{6}) = \frac{1}{2}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$ $\therefore x = \frac{\pi}{3}$ ✓ or $x = \pi$ ✓ for $0 \le x \le \pi$

Q3 (b)(i) $T = T_0 + Ae^{-kt}$ i.e. $T - T_0 = Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$ ✓

$= -k(T - T_0)$ ✓

(ii) $85 = 25 + Ae^0$ $\therefore A = 60$. ✓

$T = 25 + 60e^{-kt}$

$80 = 25 + 60e^{-k}$ ✓

$e^{-k} = \frac{55}{60}$

$-k = \ln\frac{11}{12}$ (or $k = \ln\frac{12}{11}$) ✓

$\therefore T = 25 + 60e^{(-\ln\frac{12}{11})t}$

when $t = 5$ $T = 25 + 60e^{-5\ln\frac{12}{11}}$ ✓

$\doteqdot 64$

$T \doteqdot 64°$ after a further 4 minutes. ✓

Q.4 (a) $x^2 = 4ay$ $\frac{dy}{dx} = \frac{x}{2a}$ at $x = 2ap$

$\frac{dy}{dx} = p$ ✓

$\therefore$ gradient of normal $= -\frac{1}{p}$ ✓

Eqn: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ ✓

Normal at P $\quad x+py=2ap+ap^3 \quad (1)$

Normal at Q $\quad x+qy=2aq+aq^3 \quad (2)$

(1) − (2)

$$y(p-q)=2a(p-q)+a(p^3-q^3)$$
$$y=2a+a(p^2+q^2+pq)$$
$$y=a(p^2+q^2+pq+2)$$

$\times q \quad xq+pqy=2apq+aqp^3$

$\times p \quad -xp-pqy=-2apq-apq^3$

$$x(q-p)=apq(p-q)(p+q)$$
$$x=-apq(p+q)$$

$[-apq(p+q),\ a(p^2+q^2+pq+2)] = N(X,Y)$

$$m_{PQ}=\frac{ap^2-aq^2}{2ap-2aq} \Rightarrow m_{PQ}=\frac{\cancel{a}(p+q)\cancel{(p-q)}}{2\cancel{a}\cancel{(p-q)}}=\frac{p+q}{2}$$

Chord PQ: $\quad y-ap^2=\frac{p+q}{2}(x-2ap)$

sub. $S(0,a)$: $\quad a-ap^2=\frac{p+q}{2}(-2ap)$

$a-\cancel{ap^2}=-\cancel{ap^2}-apq$

$\therefore pq=-1$ if PQ is through S.

$X=-apq(p+q) \Rightarrow X=a(p+q) \Rightarrow p+q=\frac{X}{a} \quad (3)$

$Y=a(p^2+q^2+pq+2) \Rightarrow Y=a(p^2+q^2+1) \Rightarrow p^2+q^2=\frac{Y}{a}-1 \quad (4)$

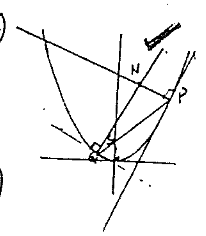
$p^2+q^2=(p+q)^2+2$

(5) $\quad \frac{Y}{a}-1=\frac{X^2}{a^2}+2$

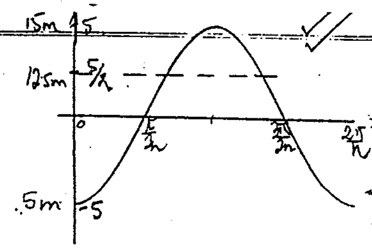
$$\frac{Y}{a}=\frac{X^2}{a^2}+3$$

$Y=\frac{X^2}{a}+3a \quad$ or $\quad X^2=a(Y-3a)$

(c)



Q5(b)



(b) $a=5$ $\frac{1}{2}$

$T=\frac{2\pi}{n}$ $\frac{1}{2}$

$\frac{25}{2}=\frac{2\pi}{n}$ $\frac{1}{2}$

$n=\frac{4\pi}{25}$ $\frac{1}{2}$

$\therefore x=-5\cos\frac{4\pi}{25}t$

when $x=\frac{5}{2}$

$$\frac{5}{2}=-5\cos\frac{4\pi}{25}t$$
$$-\frac{1}{2}=\cos\frac{4\pi}{25}t$$
$$\frac{4\pi}{25}t=\frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3},\ \ldots$$
$$t=\frac{25}{6},\ \frac{25}{3},\ \ldots$$

$\therefore$ the times between which the ship may enter the harbour are 8:40 am and 12:50 pm.

(a) (i) LHS $=\sin(\sin^{-1}x-\cos^{-1}x)$

$=\sin(\alpha-\beta)$

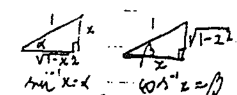
$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$=x\cdot x-\sqrt{1-x^2}\cdot\sqrt{1-x^2}$

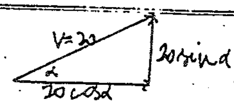
$=2x^2-1$

$=$ RHS

where



(5)

a) 

at $t=0$

$g = -10$
$\dot{x} = 20\cos\theta$ $\quad x = 0$
$\dot{y} = 20\sin\theta$ $\quad y = 0$

$\ddot{x} = 0$
$\dot{x} = 20\cos\alpha$
$x = 20t\cos\alpha$ ✓ ①

$\ddot{y} = -10$
$\dot{y} = -10t + 20\sin\alpha$
$y = -5t^2 + 20t\sin\alpha$ ✓ ②

) Sub $t = \frac{x}{20\cos\alpha} \to$ ②

$y = \frac{x\,20\sin\alpha}{20\cos\alpha} - 5\frac{x^2}{400\cos^2\alpha}$

$y = x\tan\alpha - \frac{x^2}{80}\sec^2\alpha$ ✓

) when $x = 20$, $y = h$ ✓

$\therefore h = 20\tan\alpha - \frac{400}{80}\sec^2\alpha$

$h = 20\tan\alpha - 5\sec^2\alpha$ ✓

$\frac{dh}{d\alpha} = 20\sec^2\alpha - 10\sec^2\alpha\tan\alpha$ ✓

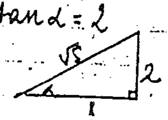
$\therefore 10\sec^2\alpha(2 - \tan\alpha) = 0$ for max

$\sec\alpha = 0$ or $\tan\alpha = 2$ ✓

$\alpha = 63°$ (nearest deg)

60°	63°	70°
+	0	−

$\therefore h$ is max. when $\tan\alpha = 2$ ✓



$h_{max} = 20\times 2 - 5\times\sqrt{5}^2$
$= 15$ metres ✓

4.1.(i)
(ii) $V = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ ✓ (b)

$= \sqrt{(20\cos\alpha)^2 + (20\sin\alpha - 10t)^2}$ ✓

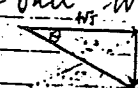
To find t:
$20 = Vt\cos\alpha$
$20 = 20t\times\frac{1}{\sqrt{5}}$
$t = \sqrt{5}$ ✓

$= \sqrt{\left(\frac{20}{\sqrt{5}}\right)^2 + \left(\frac{20}{1}\cdot\frac{2}{\sqrt{5}} - 10\times\sqrt{5}\right)^2}$

$= \sqrt{80 + (8\sqrt{5} - 10\sqrt{5})^2}$

$= \sqrt{80 + 20}$

$= 10$ m/s is the speed of ball when hits the wall on its way ↓. ✓



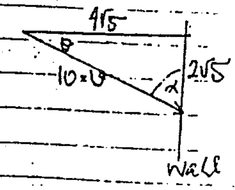
$\frac{dy}{dt} = 20\sin\alpha - 10t$
$= -2\sqrt{5}$

$\frac{dx}{dt} = 20\cos\alpha$
$= \frac{20}{\sqrt{5}}$
$= 4\sqrt{5}$

$\tan\theta = \frac{2\sqrt{5}}{4\sqrt{5}}$ ✓

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$ ✓

hits wall at



$\alpha = \tan^{-1}\left(\frac{4\sqrt{5}}{2\sqrt{5}}\right)$

$= \tan^{-1}2$ ✓